1. The key to this problem is to realize that the βs in the expression for γ have a time derivative in them. So an argument of this type is valid

$$\int_{t_1}^{t_2}\sqrt{1-\beta^2}\,dt=\int_{t_1}^{t_2}\sqrt{dt^2-\left(1/c^2\right)\left(dx^2+dy^2+dz^2\right)}.$$

But the invariance of the space-time interval shows that under a general Lorentz transformation

$$\int_{t_1}^{t_2}\sqrt{dt^2-\left(1/c^2\right)\left(dx^2+dy^2+dz^2\right)}=\int_{t'_1}^{t'_2}\sqrt{dt'^2-\left(1/c^2\right)\left(dx'^2+dy'^2+dz'^2\right)}$$

Therefore

$$\int_{t_1}^{t_2}\sqrt{1-\beta^2}\,dt=\int_{t'_1}^{t'_2}\sqrt{1-\beta'^2}\,dt'.$$

The square root differential expression is well defined because the four-velocity of any massive particle is always time-like.

A more formal and precise argument is the following. Divide the closed interval $\left[t_1,t_2\right]$ into N equal sub-intervals of duration $\Delta t=\left(t_2-t_1\right)/N$ labeled by the index i: $I_i=\left[t_1+\left(i-1\right)\Delta t,t_1+i\Delta t\right]$. By the general Lorentz transformation between frames we may establish the coordinates of the space-time events $c\left(t_1\right),\vec{x}\left(t_1\right)$ and $c\left(t_1+i\Delta t\right),\vec{x}\left(t_1+i\Delta t\right)$ in the prime frame. Call the coordinates in the K' frame $c\left(t'_0\right),\vec{x}'_0$ and $ct'_i,\vec{x}'_i$ respectively. Recall the mean value theorem from calculus

$$\int_{t_1}^{t_2}\sqrt{1-\beta^2}\,dt=\sum_{i=1}^{N}\int_{I_i}\sqrt{1-\beta^2}\,dt=\sum_{i=1}^{N}\sqrt{1-\beta^2\left(T_i\right)}\Delta t$$

for some $T_i\in I_i$. In the limit $N\to\infty$, the intervals become infinitesimals and

$$\vec{\beta}\left(T_i\right)\to\frac{\Delta\vec{x}_i}{c\Delta t}\equiv\frac{\vec{x}_i-\vec{x}_{i-1}}{c\Delta t}.$$

This means

$$\int_{t_1}^{t_2}\sqrt{1-\beta^2}\,dt=\lim_{N\to\infty}\sum_{i=1}^{N}\sqrt{\Delta t^2-\left(1/c^2\right)\left|\Delta\vec{x}_i\right|^2}$$

The Lorentz transformations are linear, and so the differentials $c\Delta t$ and $\Delta\vec{x}$ transform in the same way as ct and $\vec{x}$. Therefore, by the invariance of the space-time interval under Lorentz transformations,

$$\int_{t_1}^{t_2}\sqrt{1-\beta^2}\,dt=\lim_{N\to\infty}\sum_{i=1}^{N}\sqrt{\Delta t_i'^2-\left(1/c^2\right)\left|\Delta\vec{x}'_i\right|^2}=\int_{t'_1}^{t'_2}\sqrt{1-\beta'^2}\,dt',$$

because the durations of all the intervals in the prime frame approach zero as N increases without bound. It should be noted that $\Delta t_i'$ is NOT necessarily constant as i changes when there is acceleration in the orbit.

2. By the relativistic Lorentz Force equation

$$\vec{\mathrm{v}}\cdot\frac{d\left(\gamma m\vec{\mathrm{v}}\right)}{d\tau}=\vec{\mathrm{v}}\cdot q\left(\vec{E}+\vec{\mathrm{v}}\times\vec{B}\right)=q\vec{\mathrm{v}}\cdot\vec{E}$$

Now

$$\frac{d\gamma}{dt}=\frac{d}{dt}\frac{1}{\sqrt{1-\left|\vec{\mathrm{v}}\right|^2/c^2}}=\frac{\vec{\mathrm{v}}\cdot\left(d\vec{\mathrm{v}}/dt\right)}{c^2\left(1-\beta^2\right)^{3/2}}=\frac{\gamma^3}{c^2}\vec{\mathrm{v}}\cdot\left(d\vec{\mathrm{v}}/dt\right)$$

and

$$q\vec{\mathrm{v}}\cdot\vec{E}=\gamma\vec{\mathrm{v}}\cdot\frac{d\left(m\vec{\mathrm{v}}\right)}{dt}+\frac{m\left|\vec{\mathrm{v}}\right|^2}{c^2}\gamma^3\vec{\mathrm{v}}\cdot\left(d\vec{\mathrm{v}}/dt\right)$$
$$=\gamma m\left(1+\beta^2\gamma^2\right)\vec{\mathrm{v}}\cdot\left(d\vec{\mathrm{v}}/dt\right)=\gamma^3 m\vec{\mathrm{v}}\cdot\left(d\vec{\mathrm{v}}/dt\right).$$

So

$$\frac{d\gamma}{dt}=\frac{q\vec{\mathrm{v}}\cdot\vec{E}}{mc^2}.$$

3. Using the relativistic momentum-energy relation

$$E^2=p^2c^2+m^2c^4$$
$$2EdE=2pdpc^2$$
$$\frac{dE}{E}=\frac{\gamma\beta mcdpc^2}{\gamma^2m^2c^4}=\frac{\beta^2dp}{\gamma\beta mc}=\beta^2\frac{dp}{p}$$

4. Starting with the relativistic Lorentz force

$$\frac{d\gamma}{dt}=0\rightarrow\frac{d\left|\vec{\mathrm{v}}\right|}{dt}=0\quad\left|\vec{\mathrm{v}}\right|=const=\mathrm{v}_0$$
$$\frac{d\mathrm{v}_x}{dt}=\frac{qB}{\gamma m}\mathrm{v}_y\qquad\frac{d\mathrm{v}_y}{dt}=-\frac{qB}{\gamma m}\mathrm{v}_x$$
$$\frac{d^2\mathrm{v}_x}{dt^2}+\Omega_c^2\mathrm{v}_x=0\qquad\frac{d^2\mathrm{v}_y}{dt^2}+\Omega_c^2\mathrm{v}_y=0$$
$$\mathrm{v}_x\left(t\right)=A\cos\left(\Omega_ct+\delta\right)\qquad\mathrm{v}_y\left(t\right)=-A\sin\left(\Omega_ct+\delta\right)$$
$$\left|\vec{\mathrm{v}}\right|=A\rightarrow A=\mathrm{v}_0$$
$$x\left(t\right)=x_c+\frac{\mathrm{v}_0}{\Omega_c}\sin\left(\Omega_ct+\delta\right)\qquad y\left(t\right)=y_c+\frac{\mathrm{v}_0}{\Omega_c}\cos\left(\Omega_ct+\delta\right)$$

$$r = \frac{v_0}{\Omega_c} = \frac{\beta c}{qB/\gamma m}$$

5. The formula means $B\rho$ measured in units of T m is equal to 3.3356 times the momentum measured in GeV/c, for a singly charged particle. Note 1 T is 1 V s/m^2. So $eB\rho$ for 1 T m is 1 eV s/m, which are momentum units. 1 GeV/c is 10^9 eV/2.99792458 × 10^8 m/s = 3.33564 eV s/m. So
$$B\rho\ [\mathrm{T\ m}] = 3.33564\ p\ [\mathrm{GeV}/c]$$
For an ion with atomic weight A, the total momentum in units of GeV/c is $A[(\mathrm{GeV/u})/c]$. For a fully stripped ion, the particle has a charge of Ze. So
$$B\rho\ [\mathrm{T\ m}] = (3.33564 A/Z)\ p\ [(\mathrm{GeV/u})/c]$$
This means, for example, that the magnetic field needed to bend a fully stripped heavy ion is more than twice that needed to bend a proton at the same momentum per atomic mass unit.

6. This problem is a perfect example of the use of the magnetic rigidity. The electron relativistic momentum is $p = \sqrt{\gamma^2 - 1} m_0 c = \sqrt{\gamma^2 - 1}(0.511\ \mathrm{MeV}/c)$, the magnetic rigidity is $\sqrt{\gamma^2 - 1}(0.511\ \mathrm{MV}/c)$, and 1 T = 1 (V sec)/(m^2). Now
$$B = \frac{(B\rho)}{L} 2\sin(\theta/2)$$
for magnets in the normal configuration. The bend angles are $\pi/16 = 0.19635$ rad for the first arc and $\pi/32 = 0.098175$ rad for the rest of the arcs.

$$B_1 = \frac{\sqrt{(605/0.511)^2 - 1} \times 0.511 \times 10^6\ \mathrm{V}}{2.998 \times 10^8\ \mathrm{m/sec}(1\ \mathrm{m})} 2\sin 0.098175 = 0.3956\ \mathrm{T} = 3.956\ \mathrm{kG}$$

$$B_2 = \frac{\sqrt{(1693/0.511)^2 - 1} \times 0.511 \times 10^6\ \mathrm{V}}{2.998 \times 10^8\ \mathrm{m/sec}(1\ \mathrm{m})} 2\sin 0.04909 = 0.5542\ \mathrm{T} = 5.542\ \mathrm{kG}$$

$$B_3 = \frac{\sqrt{(2781/0.511)^2 - 1} \times 0.511 \times 10^6\ \mathrm{V}}{2.998 \times 10^8\ \mathrm{m/sec}(2\ \mathrm{m})} 2\sin 0.04909 = 0.4552\ \mathrm{T} = 4.552\ \mathrm{kG}$$

$$B_4 = \frac{\sqrt{(3868/0.511)^2 - 1} \times 0.511 \times 10^6\ \mathrm{V}}{2.998 \times 10^8\ \mathrm{m/sec}(3\ \mathrm{m})} 2\sin 0.04909 = 0.4220\ \mathrm{T} = 4.220\ \mathrm{kG}$$

$$B_5 = \frac{\sqrt{(4956/0.511)^2 - 1} \times 0.511 \times 10^6\ \mathrm{V}}{2.998 \times 10^8\ \mathrm{m/sec}(3\ \mathrm{m})} 2\sin 0.04909 = 0.5408\ \mathrm{T} = 5.408\ \mathrm{kG}$$

Arc	Electron Energy (MeV)	Number of Dipoles	Dipole Length (m)	Bend Angle (rad)	Magnetic Field (T)
1	605	16	1	0.19635	0.3956
2	1693	32	1	0.098175	0.5542
3	2781	32	2	0.098175	0.4552
4	3868	32	3	0.098175	0.4220
5	4956	32	3	0.098175	0.5408

7. Technically, we did this calculation in the lectures. If Hill's equation has focusing in the *x* direction and defocusing in the *y* direction, the equations of motion are

$$\frac{d^2x}{ds^2} + Kx = 0$$

$$\frac{d^2y}{ds^2} - Ky = 0$$

The solutions satisfying the correct boundary conditions at $s = 0$ are

$$x(s) = x_0 \cos\left(\sqrt{K}s\right) + x_0' \sin\left(\sqrt{K}s\right)/\sqrt{K}$$

$$y(s) = y_0 \cos\left(\sqrt{K}s\right) + y_0' \sin\left(\sqrt{K}s\right)/\sqrt{K}$$

Putting $s = L$ into these equations, and into these equations differentiated with respect to *s* yields the transfer matrix

$$\begin{pmatrix} x(L) \\ x'(L) \\ y(L) \\ y'(L) \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \sin\left(\sqrt{K}L\right)/\sqrt{K} & 0 & 0 \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) & 0 & 0 \\ 0 & 0 & \cosh\left(\sqrt{K}L\right) & \sinh\left(\sqrt{K}L\right)/\sqrt{K} \\ 0 & 0 & \sqrt{K}\sinh\left(\sqrt{K}L\right) & \cosh\left(\sqrt{K}L\right) \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \\ y(0) \\ y'(0) \end{pmatrix}$$

Taking the limit $L \to 0$ one obtains the thin lens approximations

$$\begin{pmatrix} x(L) \\ x'(L) \\ y(L) \\ y'(L) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/f & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \\ y(0) \\ y'(0) \end{pmatrix},$$

where the focal length f is $1/KL$.